HEDGPETH, MARIANNE AUSTIN. Linear and Circular k-Combinations with Restrictions. (1969) Directed by: Dr. Charles A. Church, Jr. pp. 21

Combinations of n distinct objects with certain restrictions are considered. The results of Kaplansky are generalized in several directions to include those of Abramson and Moser.

Included are enumerations with the following restrictions: at least s consecutive integers omitted, exactly r blocks of consecutive integers appear, and blocks of at most j consecutive integers appear. Some of these restrictions are combined and enumerated.

LINEAR AND CIRCULAR k-COMBINATIONS
WITH RESTRICTIONS

by

Marianne Austin Hedgpeth

A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
April, 1969

Approved by

C. H. Church, Jr.
Thesis Adviser

APPROVAL SHEET

This thesis has been approved by the following committee of the Faculty of the Graduate School at The University of North Carolina at Greensboro.

Thesis Adviser C. A. Church, Jr.

Oral Examination Committee Members E. E. Posey

Anna Joyce Reardon

Andrew E. Long, Jr.

April 29, 1969
Date of Examination

ACKNOWLEDGEMENT

The author's sincere appreciation goes to Dr. Charles A. Church, Jr. for invaluable assistance during the writing of this thesis.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

For several years problems involving k-combinations of n objects have filled the literature. Most of these have concerned combinations taken on a line, but selections arranged on a circle are also of interest.

The aim here is to organize and discuss in some detail these linear combinations in addition to the circular varieties. Various restrictions will be placed on the combinations. Finally, an apparently unlikely relationship will be established between k-combinations and the Fibonacci and Lucas sequences.

Any intermediate mathematics student with only a minimum of experience in the field should have little difficulty with this discussion. Familiarity with only the basic concepts is assumed. The books of Riordan [11] as well as Netto [10] are recommended references.

For notation, terminology, and basic combinatorial results, we follow Riordan.

CHAPTER II

ELEMENTARY RESULTS

Our discussion of k-combinations begins with some of the more elementary results. The basic results are well known.

The problem of placing p plus signs and q minus signs on a line with no two consecutive minuses is a simple and very useful one. This is the first problem in Riordan [11, p. 14] and we state it as follows:

THEOREM 1: <u>The number of ways of placing</u> p <u>plus signs and</u> q <u>minus signs on a line with no two consecutive minuses is</u>

$$\binom{p+1}{q};$$

<u>if arranged on a circle, so that the first and last symbols are considered consecutive, the number is</u>

$$\frac{p+q}{p}\binom{p}{q}.$$

PROOF: First place the p pluses on a line, leaving $p + 1$ slots or cells ($p - 1$ between the pluses, one before the first plus and one after the last). Now distribute the q minuses in the $p + 1$ cells in

$$\binom{p+1}{q}$$

ways. Since between any two minuses there is at least one plus sign, the first part is solved.

In the second part, the first and last symbols are considered consecutive. We have two mutually exclusive cases: (a) the first cell is chosen (the first symbol will be a minus) and (b) the first cell is not chosen (the first symbol will be a plus). In case (a), the last cell cannot be chosen, so that this resolves itself into the problem of distributing $q - 1$ minuses into $p - 1$ cells. Case (b) does not restrict the choice of the last cell, so that this becomes a problem of distributing q minuses into p cells, one to a cell. Adding the two cases, we obtain the second part of the theorem.

In his solution to the "probleme des ménages", Kaplansky [9] gives two results for restricted k-combinations. We give them here.

THEOREM 2: <u>The number of k-combinations of the first</u> n <u>natural numbers on a line with no two consecutive integers in the same combination is</u>

$$\binom{n-k+1}{k};$$

<u>if instead of being arranged on a line, the k-combinations were arranged on a circle, so that</u> n <u>and</u> 1 <u>are also consecutive, the number is</u>

$$\frac{n}{n-k}\binom{n-k}{k}.$$

PROOF: To obtain this, we observe that the k integers chosen correspond to the q minuses of Theorem 1; in each case we want no two consecutive in any choice. Therefore if $k = q$ and $n - k = p$, the desired results follow.

A frequently used fact is the following:

THEOREM 3: <u>The number of ways of putting</u> n <u>like objects into</u> m <u>different cells is</u>

$$\binom{n+m-1}{n} = \binom{n+m-1}{m-1}.$$

<u>When no cell is empty, the number is</u>

$$\binom{n-1}{m-1}.$$

PROOF: Riordan [11, p. 92] solves this problem. First we obtain the case with no cell empty. This is merely a problem of placing $m - 1$ separators between the n like objects. Since there are $n - 1$ positions in which the separators may be placed, the result is immediate.

To get the first case, place $n + m$ objects into the m different cells, no cell empty in

$$\binom{n+m-1}{m-1}$$

ways. Now remove one object from each cell; this leaves the desired n objects in the m cells and the theorem is proved.

As a generalization [5, p. 127], we restate Riordan's problem one. In how many ways can p pluses and q minuses be placed on a line with at least s pluses between any two minuses? Proceeding as before, we this time place the q minuses on a line, leaving $q + 1$ cells. In each of the $q - 1$ interior cells we put s plus signs. This leaves $p - s(q - 1)$ pluses to be distributed into the $q + 1$ cells. This we do in

$$\binom{p - sq + s + q}{q}$$

ways. With $p = n - k$ and $q = k$, we have

$$\binom{n - sk + s}{k}, \tag{1}$$

which is the number of k-combinations of the first n natural numbers such that if i occurs in a given combination, none of $i + 1, i + 2, \ldots, i + s$ can [11, p. 222].

For the circular case, either (a) there are exactly i initial pluses $(0 \leq i \leq s - 1)$, or (b) there are s or more initial pluses.

For (a), place s pluses into each of the $q - 1$ interior cells and i and $s - 1$ pluses into the first and last cell, respectively, in s ways. Distributing the remaining $p - sq$ pluses into the last q cells (excluding the first cell), we get

$$s\binom{p + q - sq - 1}{q - 1} = \frac{sq}{p + q - sq}\binom{p + q - sq}{q}$$

ways.

To obtain (b), again place s pluses into each of the $q - 1$ interior cells. Now place s pluses into the first cell, leaving the last cell empty. This leaves $p - sq$ pluses to be distributed into the $q + 1$ cells in

$$\binom{p + q - sq}{q}$$

ways. Add, and the solution is

$$\frac{p + q}{p + q - sq}\binom{p + q - sq}{q}.$$

With $k = q$, $n - k = p$, this becomes

$$\frac{n}{n - sk}\binom{n - sk}{k}, \tag{2}$$

the number of circular k-combinations such that if i appears in a combination, then none of $i + 1, i + 2, \ldots, i + s$ can [11, p. 222].

The k-combinations may be further restricted. But first the notion of a block must be introduced.

DEFINITION: A <u>block</u> is a sequence of consecutive integers which is not contained in a larger one. For example the sequence 125678 contains two blocks, one of length two and one of length four.

THEOREM 4: [5, p. 124] <u>The number of k-combinations of the first n natural numbers, on a line, with exactly r blocks of consecutive integers is</u>

$$\binom{k - 1}{r - 1}\binom{n - k + 1}{r};$$

<u>if arranged on a circle, considering</u> n <u>and</u> 1 <u>to be consecutive,</u> <u>we get</u>

$$\frac{n}{n-k}\binom{k-1}{r-1}\binom{n-k}{r}.$$

PROOF: In proving the linear case, we use the same techniques as in Theorem 2 and place the $n - k$ integers not chosen on a line, leaving the customary $n - k + 1$ cells. We now choose our r blocks from the $n - k + 1$ cells in

$$\binom{n-k+1}{r}$$

ways. We then distribute the k integers chosen into the r cells, no cell empty, in

$$\binom{k-1}{r-1}$$

ways.

The number of such circular combinations may be found similarly. This time we choose the r blocks from the $n - k + 1$ cells so that the first and last cells are not both selected. This may be done, as Theorem 2 indicates, in

$$\frac{n}{n-k}\binom{n-k}{r}$$

ways. We then proceed to distribute the k integers into the r blocks, no cell empty, as in the linear case.

Two interesting restrictions may be placed on circular k-combinations of the first n natural numbers with respect to blocks.

THEOREM 5: [5, pp. 124-125] <u>The number of circular k-combinations of the first</u> n <u>natural numbers where</u> n <u>and</u> 1 <u>never occur together is</u>

$$\frac{n - k + r}{n - k}\binom{k - 1}{r - 1}\binom{n - k}{r};$$

<u>the number when</u> n <u>and</u> 1 <u>always occur together in a block is</u>

$$\frac{k - r}{n - k}\binom{k - 1}{r - 1}\binom{n - k}{r}.$$

PROOF: To get the number of ways with n and 1 never occuring together, we observe that there are two mutually exclusive cases: (a) 1 is chosen, and (b) 1 is not chosen.

We know that 1 is chosen if and only if the first of the $n - k + 1$ cells is chosen. Since n cannot appear if 1 does, the last cell cannot be chosen. We thus pick the other $r - 1$ blocks from the remaining $n - k - 1$ cells in

$$\binom{n - k - 1}{r - 1}$$

ways. This is case (a).

If 1 is not chosen, n may be chosen. We have thus $n - k$ cells from which to select the r blocks; this is done in

$$\binom{n-k}{r}$$

ways, giving case (b).

Combining (a) and (b), we find that the r blocks may be chosen in

$$\frac{n-k+r}{n-k}\binom{n-k}{r}$$

ways.

Now distribute the k integers into the r cells, no cell empty, in

$$\binom{k-1}{r-1}$$

ways.

If, as in the second part of the theorem, 1 and n are always to occur in a block, the first and last of the $n - k + 1$ cells must be chosen. But since these cells are considered as one block, we have $r - 1$ blocks more to pick from among the remaining $n - k - 1$ cells. This is done in

$$\binom{n-k-1}{r-1}$$

ways.

Of our original $n - k + 1$ cells, we chose $r + 1$, so that we must distribute the k integers into the $r + 1$ cells in

$$\binom{k-1}{r}$$

ways. The result follows.

Abramson [1, pp. 345-346] solves an interesting problem involving k-combinations. Our next theorem shows his results.

THEOREM 6: The number of k-combinations of the first n natural numbers such that at most j consecutive integers appear in any choice is

$$A_{j+1}(n,k) = \sum_{r=0}^{\left[\frac{k}{j+1}\right]} (-1)^r \binom{n-k+1}{r} \binom{n-r(j+1)}{n-k}.$$

PROOF: Once again we observe that the method of Riordan's first problem is useful. Placing the $n - k = p$ elements not chosen on a line, we form the usual $p + 1$ cells. This time, however, we want at most j integers placed in any cell. Here a generating function is helpful. The enumerator is, assuming $j \leq q$,

$$(1 + t + t^2 + \dots + t^j)^{p+1} = \frac{(1 - t^{j+1})^{p+1}}{(1 - t)^{p+1}}$$

$$= \sum_{q=0}^{\infty} t^q \sum_{r=0}^{\left[\frac{q}{j+1}\right]} (-1)^r \binom{p+1}{r} \binom{p+q-r(j+1)}{p}.$$

Now substituting $q = k$, $p = n - k$, the coefficient of t^k is the desired result.

The case where $j = 1$ is Kaplansky's famous result (our Theorem 2).

We should thus be able to show the identity

$$\binom{n - k + 1}{k} = \sum_{r=0}^{\left[\frac{k}{2}\right]} (-1)^r \binom{n - k + 1}{r} \binom{n - 2r}{n - k} .$$

If $j = 1$, then we have the number of k-combinations with at most one consecutive integer (no two consecutive integers). The generating function used to get $A_{j+1}(n,k)$ becomes

$$(1 + t)^{p+1} = \sum_{q=0}^{p+1} \binom{p + 1}{q} t^q .$$

Alternately,

$$(1 + t)^{p+1} = \frac{(1 - t^2)^{p+1}}{(1 - t)^{p+1}}$$

$$= \sum_{q=0}^{\infty} t^q \sum_{r=0}^{\left[\frac{q}{2}\right]} (-1)^r \binom{p + 1}{r} \binom{p + q - 2r}{p} .$$

Thus we get, for $p = n - k$, $q = k$,

$$\binom{h - k + 1}{k} = \sum_{r=0}^{\left[\frac{k}{2}\right]} (-1)^r \binom{n - k + 1}{r} \binom{n - 2r}{n - k} .$$

Abramson's "Restricted Choices" suggests another result. To investigate this we must first define a gap.

DEFINITION: A gap is a block of consecutive integers, none of which appear in a given combination. For example, the combination 12568 contains one gap of length two, one gap of length one, and two gaps of length zero.

THEOREM 7: The number of k-combinations of the first n integers with gaps of length at most j is

$$A_{j+1}(n,n-k) = \sum_{r=0}^{\left[\frac{n-1}{j+1}\right]} (-1)^r \binom{1+k}{r}\binom{n-r(j+1)}{k}.$$

PROOF: This case is completely analogous to Abramson's previous result with the roles of k and $n-k$ interchanged.

Abramson [1, p. 347] suggests the following theorem:

THEOREM 8: The number of k-combinations of the first n natural numbers such that no two integers i and $i+2$ appear in any choice is

$$\sum_{s=0}^{\left[\frac{k}{2}\right]} \binom{n-2(k-1)+s}{k-s}\binom{k-s}{s}.$$

PROOF: Arrange the $n-k$ elements not chosen on a line, leaving $n-k+1$ cells. The provision of the theorem allows at most 2 elements to a cell (2 consecutive elements chosen) and no consecutive cells chosen. To get the former, divide the k elements into s groups of 2 elements each and $k-2s$ groups of one element each, a total of $k - s$ groups.

Now choose $k - s$ of the $n - k + 1$ cells, no two consecutive, in

$$\binom{n - 2k + s + 2}{k - s}$$

ways.

Of the $k - s$ cells selected, pick s of them in

$$\binom{k - s}{s}$$

ways. In each cell, place one group of two elements each. Now place one element in each of the remaining $k - 2s$ cells. Sum over s.

CHAPTER III

FIBONACCI AND LUCAS RESULTS

Let us now give further consideration to Theorem 2. Summed over k, we observe that

$$(3) \qquad F_{n+2} = \sum_{k=0}^{\left[\frac{n+1}{2}\right]} \binom{n-k+1}{k},$$

where F_n is the n^{th} Fibonacci number. The Fibonacci sequence is defined by the initial conditions $F_0 = 0$, $F_1 = 1$, and by the recurrence $F_{n+1} = F_{n-1} + F_n$.

If the second result of Theorem 2 is summed over k, we get

$$(4) \qquad L_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{k},$$

where L_n is the n^{th} Lucas number. The Lucas sequence is defined by the initial conditions $L_0 = 2$, $L_1 = 1$, and by the recurrence $L_{n+1} = L_{n-1} + L_n$.

In addition, it should be noted that if (1) and (2) are summed on k, we get, respectively, a generalized Fibonacci number and a generalized Lucas number.

With this knowledge we can consider Theorem 8 in a slightly different way. cf. [7]

THEOREM 9: For $n = 2m$, the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same combination is F^2_{m+2}. If $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$.

PROOF: The restriction that i and $i + 2$ cannot appear in any selection can be restated as (a) no two consecutive even integers appear in any selection and (b) no two consecutive odd integers appear in any selection. If the k elements chosen are composed of s of the m even integers and $k - s$ of the m odd, no two consecutive, we get

$$\sum_{s=0}^{k} \binom{m - s + 1}{s} \binom{m - (k - s) + 1}{k - s}$$

k-combinations of the first $2m$ natural numbers, where i and $i + 2$ do not appear.

To get the total number of such k-combinations, we sum on k:

$$F^2_{m+2} = \sum_{k=0}^{2\left[\frac{m+1}{2}\right]} \sum_{s=0}^{k} \binom{m - s + 1}{s} \binom{m - (k - s) + 1}{k - s} ,$$

with the usual provision that

$$\binom{a}{b} = 0 \quad \text{for} \quad b > a \geqslant 0 \ .$$

For $n = 2m + 1$, we let k be composed of s of the m even integers, no two consecutive, and $k - s$ of the $m + 1$ odd integers, no two consecutive. This gives us

$$\sum_{s=0}^{k} \binom{m - s + 1}{s} \binom{m - (k - s) + 2}{k - s}$$

k-combinations of the first $2m + 1$ natural numbers such that i and $i + 2$ do not appear. Again summing on k, we have

$$F_{m+2}F_{m+3} = \sum_{k=0}^{m+1} \sum_{s=0}^{k} \binom{m - s + 1}{s} \binom{m - (k - s) + 2}{k - s} .$$

Also of interest is the circular analogue.

THEOREM 10: <u>For</u> $n = 2m$, <u>the total number of circular k-combinations of the first</u> n <u>natural numbers such that no two elements</u> i <u>and</u> $i + 2$ <u>appear together in the same combination is</u> L_m^2; <u>if</u> $n = 2m + 1$, <u>the total is</u> $L_m L_{m+1}$.

PROOF: If $n = 2m$, 2 and $2m$ are taken to be consecutive as are 1 and $2m - 1$. Following the same pattern as before, we let the k integers chosen be s of the m even integers and $k - s$ of the m odd integers. Directly applying Theorem 2, we find

$$\sum_{s=0}^{k} \frac{m}{m - s} \binom{m - s}{s} \frac{m}{m - (k - s)} \binom{m - (k - s)}{k - s}$$

circular k-combinations such that i and $i + 2$ do not appear.

We thus get a total of

$$L_m^2 = \sum_{k=0}^{2\left[\frac{m}{2}\right]} \sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s}\frac{m}{m-(k-s)}\binom{m-(k-s)}{k-s} .$$

Similarly, we may obtain, for $n = 2m + 1$, a total of

$$L_m L_{m+1} =$$

$$\sum_{k=0}^{m} \sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s}\frac{m+1}{m-(k-s)+1}\binom{m-(k-s)+1}{k-s} .$$

We have obtained thus far a relationship between the number of linear k-combinations of the first n natural numbers such that i and $i + v$ $(v = 1,2)$ do not appear in the same combination and the Fibonacci numbers; we have also found a relationship between such circular combinations and the Lucas numbers. Our results so far bring to mind a more general problem. Brown's paper [4] suggests the following theorem.

THEOREM 11: <u>The total number of k-combinations of the first</u> n <u>natural numbers, on a line, such that no two integers</u> i <u>and</u> $i + v$ $(0 \leq v < n)$ <u>appear in any combination is</u> $F_{m+3}^{r} F_{m+2}^{v-r}$, <u>where</u> $n = mv + r$, $0 \leq r < v$.

PROOF: Divide the n elements into equivalence classes, $A_1, A_2, \ldots, A_v$, such that every element in A_j is congruent to $j \pmod v$. Then for $1 \leq j \leq r$,

$$A_j = (j,\ j + v,\ j + 2v,\ \ldots\ ,\ j + mv),$$

and for $r < j \leq v$,

$$A_j = (j,\ j + v,\ j + 2v,\ \ldots\ ,\ j + (m - 1)v).$$

It should be clear that a combination will have i and $i + v$ appearing for some i if and only if one of the A_j has two consecutive elements appearing. If $1 \leq j \leq v$, A_j has $m + 1$ elements, and there are r such A_j's. By equation (3) the total number of combinations of $m + 1$ elements, no two consecutive, in a combination is F_{m+3}.

If $r < j \leq v$, A_j has m elements and there are $v - r$ such A_j's. Likewise the total number of combinations of m elements, no two consecutive in a combination, is F_{m+2}. The result is immediate.

The case $v = 2$, $r = 0,1$ is the case of Theorem 9:

$F^0_{m+3}F^2_{m+2} = F^2_{m+2}$ if $r = 0$ (n even)

$F_{m+3}F_{m+2}$ if $r = 1$ (n odd).

The circular analogue to this theorem is now apparent [8]:

THEOREM 12: The total number of circular k-combinations of the first n natural numbers such that no two integers i and $i + v$ $(0 \leq v < n)$ appear in any combinations $L^r_{m+1}L^{v-r}_m$, where $n = mv + r$, $0 \leq r < v$.

PROOF: Divide the n elements into equivalence classes, $A_1, A_2, \ldots, A_v$, as before so that every element in A_j is congruent to $j(\text{mod } v)$. The same argument holds as before with the problem becoming this time one of counting all of the combinations of each A_j, arranged on a circle, with no two consecutive elements in any combination. By equation (4) and the preceding proof, this can be done in $L^r_{m+1}L^{v-r}_m$ ways and the problem is solved.

Considering the case where $v = 2$, we get Theorem 10. So that we have

$$L^0_{m+1}L^2_m = L^2_m \quad \text{if} \quad r = 0 \quad (n \text{ even})$$

or

$$L_{m+1}L_m \qquad \text{if} \quad r = 1 \quad (n \text{ odd}).$$

One can combine the results of the two preceding theorems to obtain results in which linear and circular restrictions are mixed. The statement of the general result is a bit messy. However, for any specific problem the solution follows easily from the methods of the respective proofs.

CHAPTER IV

SUMMARY AND CONCLUSIONS

Combinations of n objects taken k at a time might at first seem rather limited in scope. However, many problems concerning them may be investigated. We have observed that Kaplansky's results are open to a number of generalizations, especially those of Abramson and Moser. These results also lead to the Fibonacci and Lucas numbers and related summation problems.

All of these results have applications to permutations with restrictions. Some of them lead to immediate generalizations of the Fibonacci and Lucas numbers.

The restrictions are of two basic types. They are those with at least a certain number of objects omitted and those with at most a certain number of objects included.

BIBLIOGRAPHY

1. M. Abramson, "Explicit Expressions for a Class of Permutation Problems," Canad. Math. Bull., 7(1964), pp. 345-350.

2. M. Abramson, "Restricted Choices," Canad. Math. Bull., 8(1965), pp. 585-600.

3. M. Abramson and W. Moser, "Combinations, Successions, and the n-Kings Problem," Math. Magazine, 39(1966), pp. 269-273.

4. J. L. Brown, Jr., "A Combinatorial Problem Involving Fibonacci Numbers," Fibonacci Qtly., 6(1968), pp. 34-35.

5. C. A. Church, Jr., "Combinations and Successions," Math. Magazine, 41(1968), pp. 123-128.

6. C. A. Church, Jr., "On a Lemma of M. Abramson," Canad. Math. Bull., 9(1966), p. 348.

7. C. A. Church, Jr., Problem H-70, "Solution and Comments," Fibonacci Qtly., 5(1967), pp. 253-255.

8. M. Hedgpeth, "A Combinatorial Problem Involving Lucas Numbers," Fibonacci Qtly., to appear.

9. I. Kaplansky, "Solution of the 'Probleme des Menages'," Bull. Amer. Math. Soc., 49(1943), pp. 784-785.

10. E. Netto, Lehrbuch der Combinatorik, Berlin, 1927.

11. J. Riordan, An Introduction to Combinatorial Analysis, New York, 1958.